

# THE INFLUENCE OF THE TUNNEL EFFECT ON $L^\infty$ -TIME DECAY

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**ABSTRACT.** We consider the Klein-Gordon equation on a star-shaped network composed of  $n$  half-axes connected at their origins. We add a potential which is constant but different on each branch. Exploiting a spectral theoretic solution formula from a previous paper, we study the  $L^\infty$ -time decay via Hörmander's version of the stationary phase method. We analyze the coefficient  $c$  of the leading term  $c \cdot t^{-1/2}$  of the asymptotic expansion of the solution with respect to time. For two branches we prove that for an initial condition in an energy band above the threshold of tunnel effect, this coefficient tends to zero on the branch with the higher potential, as the potential difference tends to infinity. At the same time the incline to the  $t$ -axis and the aperture of the cone of  $t^{-1/2}$ -decay in the  $(t, x)$ -plane tend to zero.

## 1. INTRODUCTION

In this paper we study the  $L^\infty$ -time decay of waves in a star shaped network of one-dimensional semi-infinite media having different dispersion properties. Results in experimental physics [10, 11], theoretical physics [9] and functional analysis [5, 8] describe phenomena created in this situation by the dynamics of the tunnel effect: the delayed reflection and advanced transmission near nodes issuing two branches. Our purpose is to describe the influence of the height of a potential step on the  $L^\infty$ -time decay of wave packets above the threshold of tunnel effect, which sheds a new light on its dynamics.

In this proceedings contribution we state results for a special choice of initial conditions. The proofs in a more general context will be the core of another paper.

The dynamical problem can be described as follows:

Let  $N_1, \dots, N_n$  be  $n$  disjoint copies of  $(0, +\infty)$  with  $n \geq 2$ . Consider numbers  $a_k, c_k$  satisfying  $0 < c_k$ , for  $k = 1, \dots, n$  and  $0 \leq a_1 \leq a_2 \leq \dots \leq a_n < +\infty$ . Find a vector  $(u_1, \dots, u_n)$  of functions  $u_k : [0, +\infty) \times \overline{N_k} \rightarrow \mathbb{C}$  satisfying the Klein-Gordon equations

$$[\partial_t^2 - c_k \partial_x^2 + a_k]u_k(t, x) = 0, \quad k = 1, \dots, n,$$

on  $N_1, \dots, N_n$  coupled at zero by usual Kirchhoff conditions and complemented with initial conditions for the functions  $u_k$  and their derivatives.

Reformulating this as an abstract Cauchy problem, one is confronted with the self-adjoint operator  $A = (-c_k \cdot \partial_x^2 + a_k)_{k=1, \dots, n}$  in  $\prod_{k=1}^n L^2(N_k)$ , with a domain that incorporates the Kirchhoff transmission conditions at zero. For an exact definition of  $A$ , we refer to Section 2.

Invoking functional calculus for this operator, the solution can be given in terms of

$$e^{\pm i\sqrt{A}t}u_0 \text{ and } e^{\pm i\sqrt{A}t}v_0.$$

In a previous paper ([4], see also [3]) we construct explicitly a spectral representation of  $\prod_{k=1}^n L^2(N_k)$  with respect to  $A$  involving  $n$  families of generalized eigenfunctions. The  $k$ -th family is defined on  $[a_k, \infty)$  which reflects that  $\sigma(A) = [a_1, \infty)$  and that the multiplicity of

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the spectrum is  $j$  in  $[a_j, a_{j+1})$ ,  $j = 1, \dots, n$ , where  $a_{n+1} = +\infty$ . In this band  $(a_j, a_{j+1})$  the generalized eigenfunctions exhibit exponential decay on the branches  $N_{j+1}, \dots, N_n$ , a fact called "multiple tunnel effect" in [4].

In Section 2 we recall the solution formula proved in [4]. In Section 3 we use Hörmander's version of the stationary phase method to derive the leading term of the asymptotic expansion of the solution on certain branches and for initial conditions in a compact energy band included in  $(a_j, a_{j+1})$ . We obtain  $c \cdot t^{-1/2}$  in cones in the  $(t, x)$ -space delimited by the group velocities of the limit energies and the dependence of  $c$  on the coefficients of the operator is indicated. One can prove that outside these cones the  $L^\infty$ -norm decays at least as  $t^{-1}$ . The complete analysis will be carried out in a more detailed paper.

For the case of two branches and wave packets having a compact energy band included in  $(a_2, \infty)$ , we show in Section 4 that  $c$  tends to zero on the side of the higher potential, if  $a_1$  stays fixed and  $a_2$  tends to infinity. We observe further that the exact  $t^{-1/2}$ -decay takes place in a cone in the  $(t, x)$ -plane whose aperture and incline to the  $t$ -axis tend to zero as  $a_2$  tends to infinity. Physically the model corresponds to a relativistic particle in a one dimensional world with a potential step of amount  $a_2 - a_1$  in  $x = 0$ . Our result represents thus a dynamical feature for phenomena close to tunnel effect, which might be confirmed by physical experiments.

Our results are designed to serve as tools in some pertinent applications as the study of more general networks of wave guides (for example microwave networks [17]) and the treatment of coupled transmission conditions [7].

For the Klein-Gordon equation in  $\mathbb{R}^n$  with constant coefficients the  $L^\infty$ -time decay  $c \cdot t^{-1/2}$  has been proved in [15]. Adapting their method to a spectral theoretic solution formula for two branches, it has been shown in [1, 2] that the  $L^\infty$ -norm decays at least as  $c \cdot t^{-1/4}$ .

In [14] and several related articles, the author studies the  $L^\infty$ -time decay for crystal optics using similar methods.

In [13], the authors consider general networks with semi-infinite ends. They give a construction to compute some generalized eigenfunctions but no attempt is made to construct explicit inversion formulas. In [6] the relation of the eigenvalues of the Laplacian in an  $L^\infty$ -setting on infinite, locally finite networks to the adjacency operator of the network is studied.

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## 2. A SOLUTION FORMULA

The aim of this section is to recall the tools we used in [4] as well as the solution formula of the same paper for a special initial condition and to adapt this formula for the use of the stationary phase method in the next section.

**Definition 2.1** (Functional analytic framework).

- i) Let  $n \geq 2$  and  $N_1, \dots, N_n$  be  $n$  disjoint sets identified with  $(0, +\infty)$ . Put  $N := \bigcup_{k=1}^n \overline{N_k}$ , identifying the endpoints 0.

For the notation of functions two viewpoints are useful:

- functions  $f$  on the object  $N$  and  $f_k$  is the restriction of  $f$  to  $N_k$ .
- $n$ -tuples of functions on the branches  $N_k$ ; then sometimes we write  $f = (f_1, \dots, f_n)$ .

- ii) Two transmission conditions are introduced:

$$(T_0): (u_k)_{k=1, \dots, n} \in \prod_{k=1}^n C(\overline{N_k}) \text{ satisfies } u_i(0) = u_k(0), \ i, k \in \{1, \dots, n\}.$$

This condition in particular implies that  $(u_k)_{k=1,\dots,n}$  may be viewed as a well-defined function on  $N$ .

$$(T_1): (u_k)_{k=1,\dots,n} \in \prod_{k=1}^n C^1(\overline{N_k}) \text{ satisfies } \sum_{k=1}^n c_k \cdot \partial_x u_k(0^+) = 0.$$

iii) Define the real Hilbert space  $H = \prod_{k=1}^n L^2(N_k)$  with scalar product

$$(u, v)_H = \sum_{k=1}^n (u_k, v_k)_{L^2(N_k)}$$

and the operator  $A : D(A) \longrightarrow H$  by

$$D(A) = \left\{ (u_k)_{k=1,\dots,n} \in \prod_{k=1}^n H^2(N_k) : (u_k)_{k=1,\dots,n} \text{ satisfies } (T_0) \text{ and } (T_1) \right\},$$

$$A((u_k)_{k=1,\dots,n}) = (A_k u_k)_{k=1,\dots,n} = (-c_k \cdot \partial_x^2 u_k + a_k u_k)_{k=1,\dots,n}.$$

Note that, if  $c_k = 1$  and  $a_k = 0$  for every  $k \in \{1, \dots, n\}$ ,  $A$  is the Laplacian in the sense of the existing literature, cf. [6, 13].

**Definition 2.2** (Fourier-type transform  $V$ ).

i) For  $k \in \{1, \dots, n\}$  and  $\lambda \in \mathbb{C}$  let

$$\xi_k(\lambda) := \sqrt{\frac{\lambda - a_k}{c_k}} \quad \text{and} \quad s_k := -\frac{\sum_{l \neq k} c_l \xi_l(\lambda)}{c_k \xi_k(\lambda)}.$$

Here, and in all what follows, the complex square root is chosen in such a way that  $\sqrt{r \cdot e^{i\phi}} = \sqrt{r} e^{i\phi/2}$  with  $r > 0$  and  $\phi \in [-\pi, \pi)$ .

ii) For  $\lambda \in \mathbb{C}$  and  $j, k \in \{1, \dots, n\}$ , we define generalized eigenfunctions  $F_\lambda^{\pm, j} : N \rightarrow \mathbb{C}$  of  $A$  by  $F_\lambda^{\pm, j}(x) := F_{\lambda, k}^{\pm, j}(x)$  with

$$\begin{cases} F_{\lambda, k}^{\pm, j}(x) = \cos(\xi_j(\lambda)x) \pm i s_j(\lambda) \sin(\xi_j(\lambda)x), & \text{for } k = j, \\ F_{\lambda, k}^{\pm, j}(x) = \exp(\pm i \xi_k(\lambda)x), & \text{for } k \neq j. \end{cases}$$

for  $x \in \overline{N_k}$ .

iii) For  $l = 1, \dots, n$  let

$$q_l(\lambda) := \begin{cases} 0, & \text{if } \lambda < a_l, \\ \frac{c_l \xi_l(\lambda)}{|\sum_{j=1}^n c_j \xi_j(\lambda)|^2}, & \text{if } a_l < \lambda. \end{cases}$$

iv) Considering for every  $k = 1, \dots, n$  the weighted space  $L^2((a_k, +\infty), q_k)$ , we set  $L_q^2 := \prod_{k=1}^n L^2((a_k, +\infty), q_k)$ . The corresponding scalar product is

$$(F, G)_q := \sum_{k=1}^n \int_{(a_k, +\infty)} q_k(\lambda) F_k(\lambda) \overline{G_k(\lambda)} d\lambda$$

and its associated norm  $|F|_q := (F, F)_q^{1/2}$ .

v) For all  $f \in L^1(N, \mathbb{C})$  we define  $Vf : \prod_{k=1}^n [a_k, +\infty) \rightarrow \mathbb{C}$  by

$$(Vf)_k(\lambda) := \int_N f(x) \overline{(F_\lambda^{-, k})}(x) dx, \quad k = 1, \dots, n.$$

In [4], we show that  $V$  diagonalizes  $A$  and we determine a metric setting in which it is an isometry. Let us recall these useful properties of  $V$  as well as the fact that the property  $u \in D(A^j)$  can be characterized in terms of the decay rate of the components of  $Vu$ .

**Theorem 2.3.** *Endow  $\prod_{k=1}^n C_c^\infty(N_k)$  with the norm of  $H = \prod_{k=1}^n L^2(N_k)$ . Then*

- i)  $V : \prod_{k=1}^n C_c^\infty(N_k) \rightarrow L_q^2$  is isometric and can be extended to an isometry  $\tilde{V} : H \rightarrow L_q^2$ , which we shall again denote by  $V$  in the following.
- ii)  $V : H \rightarrow L_q^2$  is a spectral representation of  $H$  with respect to  $A$ . In particular,  $V$  is surjective.
- iii) The spectrum of the operator  $A$  is  $\sigma(A) = [a_1, +\infty)$ .
- iv) For  $l \in \mathbb{N}$  the following statements are equivalent:
  - (a)  $u \in D(A^l)$ ,
  - (b)  $\lambda \mapsto \lambda^l (Vu)(\lambda) \in L_q^2$ ,
  - (c)  $\lambda \mapsto \lambda^l (Vu)_k(\lambda) \in L^2((a_k, +\infty), q_k)$ ,  $k = 1, \dots, n$ .

Denoting  $F_\lambda(x) := (F_\lambda^{-,1}(x), \dots, F_\lambda^{-,n}(x))^T$  and  $P_j = \begin{pmatrix} I_j & 0 \\ 0 & 0 \end{pmatrix}$ , where  $I_j$  is the  $j \times j$  identity matrix, for  $\lambda \in (a_j, a_{j+1})$  it holds:  $F_\lambda^T q(\lambda) F_\lambda = (P_j F_\lambda)^T q(\lambda) (P_j F_\lambda)$  and

$$P_j F_\lambda = \begin{pmatrix} (+, *, *, \dots, *, e^{-|\xi_{j+1}|x}, \dots, e^{-|\xi_n|x}) \\ (*, +, *, \dots, *, e^{-|\xi_{j+1}|x}, \dots, e^{-|\xi_n|x}) \\ (*, *, +, \dots, *, e^{-|\xi_{j+1}|x}, \dots, e^{-|\xi_n|x}) \\ \vdots \\ (*, *, \dots, *, +, e^{-|\xi_{j+1}|x}, \dots, e^{-|\xi_n|x}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (1)$$

Here  $*$  means  $e^{-i\xi_k(\lambda)x}$  and  $+$  means  $\cos(\xi_k(\lambda)x) - is_k(\lambda)\sin(\xi_k(\lambda)x)$  in the  $k$ -th column for  $k = 1, \dots, j$ . This can be interpreted as a multiple tunnel effect (tunnel effect in the last  $(n-j)$  branches with different exponential decay rates). For  $\lambda$  near  $a_{j+1}$ , the exponential decay of the function  $x \mapsto e^{-|\xi_{j+1}|x}$  is slow. The tunnel effect is weaker on the other branches since the exponential decay is quicker.

We are now interested in the Abstract Cauchy Problem

$$(ACP) : u_{tt}(t) + Au(t) = 0, \quad t > 0, \quad \text{with } u(0) = u_0, \quad u_t(0) = 0.$$

By the surjectivity of  $V$  (cf. Theorem 2.3 (ii)) for every  $j, k \in \{1, \dots, n\}$  with  $k \leq j$  there exists an initial condition  $u_0 \in H$  satisfying

**Condition**  $(A_{j,k})$ :  $(Vu_0)_l \equiv 0$ ,  $l \neq k$  and  $(Vu_0)_k \in C_c^2((a_j, a_{j+1}))$ .

*Remark 2.4.* i) We use the convention  $a_{n+1} = +\infty$ .

ii) For  $u_0$  satisfying  $(A_{j,k})$  there exist  $a_j < \lambda_{\min} < \lambda_{\max} < a_{j+1}$  such that

$$\text{supp}(Vu_0)_k \subset [\lambda_{\min}, \lambda_{\max}]$$

iii) If  $u_0 \in H$  satisfies  $(A_{j,k})$ , then  $u_0 \in D(A^\infty) = \bigcap_{l \geq 0} D(A^l)$ , due to Theorem 2.3 (iv), since

$\lambda \mapsto \lambda^l (Vu)_m(\lambda) \in L^2((a_m, +\infty), q_m)$ ,  $m = 1, \dots, n$  for all  $l \in \mathbb{N}$  by the compactness of  $\text{supp}(Vu_0)_m$ .

**Theorem 2.5** (Solution formula of (ACP) in a special case). *Fix  $j, k \in \{1, \dots, n\}$  with  $k \leq j$ . Suppose that  $u_0$  satisfies Condition  $(A_{j,k})$ . Then there exists a unique solution  $u$  of (ACP) with  $u \in C^l([0; +\infty), D(A^{m/2}))$  for all  $l, m \in \mathbb{N}$ . For  $x \in N_r$  with  $r \leq j$  such that  $r \neq k$  and  $t \geq 0$ , we have the representation*

$$u(t, x) = \frac{1}{2}(u_+(t, x) + u_-(t, x))$$

with

$$u_\pm(t, x) := \int_{\lambda_{\min}}^{\lambda_{\max}} e^{\pm i\sqrt{\lambda}t} q_k(\lambda) e^{-i\xi_r(\lambda)x} (Vu_0)_k(\lambda) d\lambda, \quad (2)$$

*Proof.* Since  $v_0 = u_t(0) = 0$ , we have for the solution of (ACP) the representation

$$u(t) = V^{-1} \cos(\sqrt{\lambda}t) Vu_0.$$

(cf. for example [1, Theorem 5.1]). The expression for  $V^{-1}$  given in [4] yields the formula for  $u_\pm$ .  $\square$

*Remark 2.6.* Expression (2) comes from a term of the type  $*$  in  $F_\lambda$  (see (1)) via the representation of  $V^{-1}$ . A solution formula for arbitrary initial conditions which is valid on all branches is available in [4]. This general expression is not needed in the following.

### 3. $L^\infty$ -TIME DECAY

The time asymptotics of the  $L^\infty$ -norm of the solution of hyperbolic problems is an important qualitative feature, for example in view of the study of nonlinear perturbations.

In [16] the author derives the spectral theory for the 3D-wave equation with different propagation speeds in two adjacent wedges. Further he attempts to give the  $L^\infty$ -time decay which he reduces to a 1D-Klein-Gordon problem with potential step (with a frequency parameter). He uses interesting tools, but his argument is technically incomplete: the backsubstitution (see the proof of Theorem 3.2 below) has not been carried out, and thus his results cannot be reliable. Nevertheless, we have been inspired by some of his techniques.

The main problem to determine the  $L^\infty$ -norm is the oscillatory nature of the integrands in the solution formula (2). The stationary phase formula as given by L. Hörmander in Theorem 7.7.5 of [12] provides a powerful tool to treat this situation.

In the following Theorem we formulate a special case of this result relevant for us.

**Theorem 3.1** (Stationary phase method). *Let  $K$  be a compact interval in  $\mathbb{R}$ ,  $X$  an open neighborhood of  $K$ . Let  $U \in C_0^2(K)$ ,  $\Psi \in C^4(X)$  and  $\text{Im}\Psi \geq 0$  in  $X$ . If there exists  $p_0 \in X$  such that  $\frac{\partial}{\partial p}\Psi(p_0) = 0$ ,  $\frac{\partial^2}{\partial p^2}\Psi(p_0) \neq 0$ , and  $\text{Im}\Psi(p_0) = 0$ ,  $\frac{\partial}{\partial p}\Psi(p) \neq 0$ ,  $p \in K \setminus \{p_0\}$ , then*

$$\left| \int_K U(p) e^{i\omega\Psi(p)} dp - e^{i\omega\Psi(p_0)} \left[ \frac{\omega}{2\pi i} \frac{\partial^2}{\partial p^2} \Psi(p_0) \right]^{-1/2} U(p_0) \right| \leq C(K) \|U\|_{C^2(K)} \omega^{-1}.$$

for all  $\omega > 0$ . Moreover  $C(K)$  is bounded when  $\Psi$  stays in a bounded set in  $C^4(X)$ .

**Theorem 3.2** (Time-decay of the solution of (ACP) in a special case). *Fix  $j, k \in \{1, \dots, n\}$  with  $k \leq j$ . Suppose that  $u_0$  satisfies Condition  $(A_{j,k})$  and choose  $\lambda_{\min}, \lambda_{\max} \in (a_j, a_{j+1})$  such that*

$$\text{supp}(Vu_0)_k \subset [\lambda_{\min}, \lambda_{\max}] \subset (a_j, a_{j+1}).$$

*Then for all  $x \in N_r$  with  $r \leq j$  and  $r \neq k$  and all  $t \in \mathbb{R}^+$  such that  $(t, x)$  lies in the cone described by*

$$\sqrt{\frac{\lambda_{\max}}{c_r(\lambda_{\max} - a_r)}} \leq \frac{t}{x} \leq \sqrt{\frac{\lambda_{\min}}{c_r(\lambda_{\min} - a_r)}}, \quad (3)$$

there exists  $H(t, x, u_0) \in \mathbb{C}$  and a constant  $c(u_0)$  satisfying

$$\left| u_+(t, x) - H(t, x, u_0)t^{-1/2} \right| \leq c(u_0) \cdot t^{-1}, \quad (4)$$

where  $u_+$  is defined in Theorem 2.5, with

$$|H(t, x, u_0)| \leq \left( \frac{2\pi c_k}{c_r} \right)^{1/2} \lambda_{\max}^{3/4} \cdot \frac{\max_{v \in [v_{\min}, v_{\max}]} \sqrt{|(a_r - a_k)v + a_r|}}{\left( \sum_{l \leq r} \sqrt{c_l} \sqrt{(a_r - a_l)v_{\min} + a_r} \right)^2} \cdot \|(Vu_0)_k\|_{\infty},$$

where  $v_{\min} := \frac{a_r}{\lambda_{\max} - a_r}$  and  $v_{\max} := \frac{a_r}{\lambda_{\min} - a_r}$ .

*Remark 3.3.* i) Note that (3) is equivalent to

$$v_{\min} \leq v(t, x) := c_r(t/x)^2 - 1 \leq v_{\max}. \quad (5)$$

- ii) The hypotheses of Theorem 3.2 imply that  $j \geq 2$ .
- iii) An explicit expression for  $H(t, x, u_0)$  is given at the end of the proof in (8).
- iv) We have chosen to investigate only  $u_+$  in this proceedings article, since the expression for  $u_-$  does not possess a stationary point in its phase. Hence, one can prove that its contribution will decay at least as  $ct^{-1}$ . A detailed analysis will follow in a forthcoming paper.

*Proof.* We devide the proof in five steps.

*First step: Substitution.* Realizing the substitution  $p := \xi_r(\lambda) = \sqrt{\frac{\lambda - a_r}{c_r}}$  in the expression for  $u_+$  given in Theorem 2.5 leads to:

$$u_+(t, x) = 2c_r \int_{p_{\min}}^{p_{\max}} e^{i\sqrt{a_r + c_r p^2}t} q_k(a_r + c_r p^2) e^{-ipx} (Vu_0)_k(a_r + c_r p^2) p \, dp$$

with  $p_{\min} := \xi_r(\lambda_{\min})$  and  $p_{\max} := \xi_r(\lambda_{\max})$ .

*Second step: Change of the parameters  $(t, x)$ .* In order to get bounded parameters, we change  $(t, x)$  into  $(\tau, \chi)$  defined by

$$\tau = \frac{t}{\omega} \quad \text{and} \quad \chi = \frac{x}{\omega} \quad \text{with} \quad \omega = \sqrt{t^2 + x^2},$$

following an argument from [16]. Thus the argument of the exponential in the integral defining  $u_+$  becomes:

$$i\omega(\sqrt{a_r + c_r p^2}\tau - p\chi) =: i\omega\varphi(p, \tau, \chi).$$

Note that  $\tau, \chi \in [0, 1]$  for  $t, x \in [0, \infty)$ .

*Third step: Application of the stationary phase method.* Now we want to apply Theorem 3.1 to  $u_+$  with the amplitude  $U$  and the phase  $\Psi$  defined by:

$$U(p) := q_k(a_r + c_r p^2)(Vu_0)_k(a_r + c_r p^2)p, \quad \Psi(p) := \varphi(p, \tau, \chi), \quad p \in [p_{\min}, p_{\max}].$$

The functions  $U$  and  $\Psi$  satisfy the regularity conditions on the compact interval  $K := [p_{\min}, p_{\max}]$  and  $\Psi$  is a real-valued function. One easily verifies, that for  $\tau \neq 0$

$$\Psi'(p) = \frac{c_r p}{\sqrt{a_r + c_r p^2}} \tau - \chi = 0 \iff p = p_0 := \sqrt{\frac{a_r \chi^2}{c_r(c_r \tau^2 - \chi^2)}} = \sqrt{\frac{a_r x^2}{c_r(c_r t^2 - x^2)}},$$

and that this stationary point  $p_0$  belongs to the interval of integration  $[p_{\min}, p_{\max}]$ , if and only if  $(t, x)$  lies in the cone defined by (3). Furthermore for  $p \in \mathbb{R}$

$$\frac{\partial^2 \Psi}{\partial p^2}(p) = \frac{\partial^2 \varphi}{\partial p^2}(p, \tau, \chi) = \tau \frac{c_r a_r}{(a_r + c_r p^2)^{3/2}} \neq 0.$$

Thus, Theorem 3.1 implies that for all  $(t, x)$  satisfying (3) there exists a constant  $C(K, \tau, \chi) > 0$  such that

$$\left| u_+(t, x) - e^{-i\omega\varphi(p_0, \tau, \chi)} \underbrace{\left( \frac{\omega}{2i\pi} \frac{\partial^2 \varphi}{\partial p^2}(p_0, \tau, \chi) \right)^{-1/2}}_{(*)} U(p_0) \right| \leq C(K, \tau, \chi) \|U\|_{C^2(K)} \omega^{-1}$$

for all  $\omega > 0$ .

*Fourth step: Backsubstitution.* We must now control the dependence of  $C(K, \tau, \chi)$  on the parameters  $\tau, \chi$ . To this end, one has to assure that  $\Psi = \varphi(\cdot, \tau, \chi)$  stays in a bounded set in  $C^4(X)$ , if  $\tau$  and  $\chi$  vary in  $[0, 1]$ , where we choose  $X = (p_m, p_M)$  such that  $0 < p_m < p_{\min} < p_{\max} < p_M < \infty$ . This follows using the above expressions for  $\frac{\partial \varphi}{\partial p}$ ,  $\frac{\partial^2 \varphi}{\partial p^2}$  and

$$\frac{\partial^3 \varphi}{\partial p^3}(p, \tau, \chi) = -\frac{3a_r c_r^2 p}{(a_r + c_r p^2)^{5/2}} \tau, \quad \frac{\partial^4 \varphi}{\partial p^4}(p, \tau, \chi) = -\frac{3a_r c_r^2 (a_r - 4p^2 c_r)}{(a_r + c_r p^2)^{7/2}} \tau$$

for  $p \in X$ . Thus Theorem 3.1 implies that there exists a constant  $C(K) > 0$  such that  $C(K, \tau, \chi) \leq C(K)$  for all  $\tau, \chi \in [0, 1]$ .

To evaluate  $(*)$  we observe that  $p_0 = \frac{1}{\sqrt{c_r}} \sqrt{\frac{a_r}{c_r(t/x)^2 - 1}}$ . This implies

$$\xi_l(a_r + c_r p_0^2) = \sqrt{\frac{(a_r + c_r p_0^2) - a_l}{c_l}} = \frac{1}{\sqrt{c_l}} \frac{\sqrt{(a_r - a_l)(c_r(t/x)^2 - 1) + a_r}}{\sqrt{c_r(t/x)^2 - 1}}$$

and thus

$$\begin{aligned} q_k(a_r + c_r p_0^2) &= \frac{c_k \xi_k(a_r + c_r p_0^2)}{|\sum_{l=1}^n c_l \xi_l(a_r + c_r p_0^2)|^2} \\ &= \sqrt{c_k} \sqrt{c_r(t/x)^2 - 1} \frac{\sqrt{(a_r - a_k)(c_r(t/x)^2 - 1) + a_r}}{\left| \sum_{l=1}^n \sqrt{c_l} \sqrt{(a_r - a_l)(c_r(t/x)^2 - 1) + a_r} \right|^2}. \end{aligned}$$

Finally,

$$\frac{\partial^2 \varphi}{\partial p^2}(p_0, \tau, \chi) = \tau \frac{c_r a_r}{(a_r + c_r p_0^2)^{3/2}} = \tau \frac{(c_r \tau^2 - \chi^2)^{3/2}}{(a_r c_r)^{1/2} \tau^2} = \tau (c_r a_r)^{-1/2} \left( \frac{c_r(t/x)^2 - 1}{(t/x)^2} \right)^{3/2}.$$

Combining these results and using  $\omega\tau = t$  we find

$$(*) = \left( \frac{\omega}{2i\pi} \frac{\partial^2 \varphi}{\partial p^2}(p_0, \tau, \chi) \right)^{-1/2} q_k(a_r + c_r p_0^2) p_0 (V u_0)_k(a_r + c_r p_0^2)$$

$$\begin{aligned}
&= (2i\pi)^{1/2} t^{-1/2} (c_r a_r)^{1/4} \left( \frac{(t/x)^2}{c_r (t/x)^2 - 1} \right)^{3/4} \times \\
&\quad \times \sqrt{c_k} \sqrt{c_r (t/x)^2 - 1} \frac{\sqrt{(a_r - a_k) (c_r (t/x)^2 - 1) + a_r}}{\left| \sum_l \sqrt{c_l} \sqrt{(a_r - a_l) (c_r (t/x)^2 - 1) + a_r} \right|^2} \times \\
&\quad \times \frac{1}{\sqrt{c_r}} \sqrt{\frac{a_r}{c_r (t/x)^2 - 1}} (V u_0)_k (a_r + c_r p_0^2) \\
&= (2i\pi)^{1/2} a_r^{3/4} c_r^{1/4} h_1(t, x) h_2(t, x) (V u_0)_k (a_r + c_r p_0^2) t^{-1/2}
\end{aligned}$$

with

$$h_1(t, x) := \left( \frac{(t/x)^2}{c_r (t/x)^2 - 1} \right)^{3/4}, \quad h_2(t, x) := \frac{\sqrt{(a_r - a_k) (c_r (t/x)^2 - 1) + a_r}}{\left| \sum_l \sqrt{c_l} \sqrt{(a_r - a_l) (c_r (t/x)^2 - 1) + a_r} \right|^2}$$

*Fifth step: Uniform estimates.* It remains to estimate  $h_1(t, x) h_2(t, x) (V u_0)_k (a_r + c_r p_0^2)$  uniformly in  $t$  and  $x$ , if  $(t, x)$  satisfies (3). To this end we note that the function  $b \mapsto \frac{b}{c_r b - 1}$  is a decreasing function on  $(1/c_r, +\infty)$ . Thus the maximum of  $h_1$  for  $(t, x)$  satisfying (3) is attained at  $\frac{t}{x} = \sqrt{\frac{\lambda_{\max}}{c_r (\lambda_{\max} - a_r)}}$ . This implies

$$h_1(t, x) \leq \left( \frac{1}{c_r a_r} \lambda_{\max} \right)^{3/4} \text{ for } (t, x) \text{ satisfying (3).} \quad (6)$$

Let us now estimate  $h_2$ . For a fixed  $v$  in  $[v_{\min}, v_{\max}]$ , we denote by  $I_1$  the set of indices  $l$  such that  $(a_r - a_l)v + a_r \geq 0$  and  $I_2 = \{1, \dots, n\} \setminus I_1$ .

Then, for any  $v$  in  $[v_{\min}, v_{\max}]$  we have  $\{1, \dots, r\} \subset I_1$  (since  $v_{\min} > 0$ ) and

$$\begin{aligned}
&\left| \sum_l \sqrt{c_l} \sqrt{(a_r - a_l)v + a_r} \right|^2 \\
&= \left( \sum_{l \in I_1} \sqrt{c_l} \sqrt{(a_r - a_l)v + a_r} \right)^2 + \left| \sum_{l \in I_2} \sqrt{c_l} \sqrt{(a_r - a_l)v + a_r} \right|^2 \\
&\geq \left( \sum_{l \leq r} \sqrt{c_l} \sqrt{(a_r - a_l)v + a_r} \right)^2 \\
&\geq \left( \sum_{l \leq r} \sqrt{c_l} \sqrt{(a_r - a_l)v_{\min} + a_r} \right)^2.
\end{aligned}$$

Thus, (5) implies

$$|h_2(t, x)| \leq \frac{\max_{v \in [v_{\min}, v_{\max}]} \sqrt{|(a_r - a_k)v + a_r|}}{\left( \sum_{l \leq r} \sqrt{c_l} \sqrt{(a_r - a_l)v_{\min} + a_r} \right)^2}. \quad (7)$$

Putting everything together, the assertion of the theorem is valid for

$$H(t, x, u_0) := e^{-i\varphi(p_0, t, x)} (2i\pi)^{1/2} a_r^{3/4} c_r^{1/4} c_k^{1/2} h_1(t, x) h_2(t, x) (V u_0)_k (a_r + c_r p_0^2). \quad (8)$$

Finally the right hand side of estimate (4) is derived from the inequality

$$C(K, \tau, \chi) \|U\|_{C^2(K)} \omega^{-1} \leq C(K) \|p \mapsto q_k(a_r + c_r p^2) (V u_0)_k (a_r + c_r p^2) p\|_{C^2([p_{\min}, p_{\max}])} t^{-1} \quad (9)$$

The  $C^2$ -norm is finite, since the involved functions are regular on the compact set  $[p_{\min}, p_{\max}]$ .  $\square$



## 4. GROWING POTENTIAL STEP

For this section we specialize to the case of two branches  $N_1$  and  $N_2$  and, for the sake of simplicity, we also set  $c_1 = c_2 = 1$ . We show that, choosing a generic initial condition  $u_0$  in a compact energy band included in  $(a_2, \infty)$ , the coefficient  $H(t, x, u_0)$  in the asymptotic expansion of Theorem 3.2 tends to zero, if the potential step  $a_2 - a_1$  tends to infinity. Simultaneously the cone of the exact  $t^{-1/2}$ -decay shrinks and inclines toward the  $t$ -axis.

**Theorem 4.1.** *Let  $0 < \alpha < \beta < 1$  and  $\psi \in C_c^2((\alpha, \beta))$  with  $\|\psi\|_\infty = 1$  be given. Setting  $\tilde{\psi}(\lambda) := \psi(\lambda - a_2)$ , we choose the initial condition  $u_0 \in H$  satisfying  $(Vu_0)_2 \equiv 0$  and  $(Vu_0)_1 = \tilde{\psi}$ . Furthermore, let  $u_+$  be defined as in Theorem 2.5.*

*Then there is a constant  $C(\psi, \alpha, \beta)$  independent of  $a_1$  and  $a_2$ , such that for all  $t \in \mathbb{R}^+$  and all  $x \in N_2$  with*

$$\sqrt{\frac{a_2 + \beta}{\beta}} \leq \frac{t}{x} \leq \sqrt{\frac{a_2 + \alpha}{\alpha}}$$

*the value  $H(t, x, u_0)$  given in (8) satisfies*

$$|u_+(t, x) - H(t, x, u_0) \cdot t^{-1/2}| \leq C(\psi, \alpha, \beta) \cdot t^{-1}$$

*and*

$$|H(t, x, u_0)| \leq \sqrt{2\pi} \frac{\sqrt{\beta}(a_2 + \beta)^{3/4}}{\sqrt{a_2}\sqrt{a_2 - a_1 + \beta}}.$$

*Proof.* Note that it is always possible to choose the initial condition in the indicated way, thanks to the surjectivity of  $V$ , cf. Theorem 2.3 ii).

The constant  $C(\psi, \alpha, \beta)$  has been already calculated in Theorem 3.2. It remains to make sure that it is independent of  $a_1$  and  $a_2$  and to prove the estimate for  $|H(t, x, u_0)|$ .

We start with the latter and carry out a refined analysis of the proof of Theorem 3.2 for our special situation. Using the notation of this proof, (8) yields

$$|H(t, x, u_0)| = \sqrt{2\pi} a_2^{3/4} h_1(t, x) |h_2(t, x)| \cdot \|(Vu_0)_1\|_\infty.$$

By (6) and  $\lambda_{\max} = a_2 + \beta$  we find

$$h_1(t, x) \leq \frac{(a_2 + \beta)^{3/4}}{a_2^{3/4}}$$

and, investing the definition of  $h_2$  together with (5), we have

$$\begin{aligned} |h_2(t, x)| &= \left| \frac{\sqrt{(a_2 - a_1)((t/x)^2 - 1) + a_2}}{(\sqrt{(a_2 - a_1)((t/x)^2 - 1) + a_2} + \sqrt{a_2})^2} \right| \leq \frac{1}{\sqrt{(a_2 - a_1)((t/x)^2 - 1) + a_2}} \\ &\leq \frac{1}{\sqrt{(a_2 - a_1)v_{\min} + a_2}}. \end{aligned}$$

Putting in the definitions of  $v_{\min}$  and afterwards  $\lambda_{\max}$  and rearranging terms, this leads to

$$|h_2(t, x)| \leq \frac{\sqrt{\beta}}{\sqrt{a_2}\sqrt{a_2 - a_1 + \beta}}.$$

Since  $\|(Vu_0)_1\|_\infty = \|\tilde{\psi}\|_\infty = \|\psi\|_\infty$  was set to 1, we arrive at the estimate

$$|H(t, x, u_0)| \leq \sqrt{2\pi} (a_2 + \beta)^{3/4} \frac{\sqrt{\beta}}{\sqrt{a_2}\sqrt{a_2 - a_1 + \beta}}.$$

Going again back to Theorem 3.2 for the constant  $C$  we have by (9)

$$C = C(K) \|U(p)\|_{C^2(K)},$$

where

$$U(p) = pq_1(a_2 + p^2)(Vu_0)_1(a_2 + p^2), \quad p \in K,$$

and

$$K = [p_{\min}, p_{\max}] = [\xi_2(a_2 + \alpha), \xi_2(a_2 + \beta)] = [\sqrt{\alpha}, \sqrt{\beta}].$$

Thus, the constant  $C(K)$  is independent of  $a_1$  and  $a_2$  and we can start to estimate the  $C^2$ -norm of  $U$ :

$$\begin{aligned} U(p) &= p\tilde{\psi}(a_2 + p^2) \frac{\xi_1(a_2 + p^2)}{|\xi_1(a_2 + p^2) + \xi_2(a_2 + p^2)|^2} = p\psi(p^2) \frac{\sqrt{a_2 - a_1 + p^2}}{(\sqrt{a_2 - a_1 + p^2} + p)^2} \\ &= p\psi(p^2) \frac{f(p)}{(f(p) + p)^2} \end{aligned}$$

where  $f(p) := \sqrt{a_2 - a_1 + p^2}$ . For the function  $U$  itself we find

$$|U(p)| \leq \sqrt{\beta} \|\psi\|_{\infty} \frac{1}{f(p)} = \frac{\sqrt{\beta}}{\sqrt{a_2 - a_1 + p^2}} \leq \frac{\sqrt{\beta}}{\sqrt{a_2 - a_1 + \alpha}} \leq \frac{\sqrt{\beta}}{\sqrt{\alpha}}.$$

Calculating the derivatives is lengthy, but using  $f'(p)f(p) = p$ , one finds constants  $C_1$  and  $C_2$  depending only on  $\psi$ ,  $\alpha$  and  $\beta$  with

$$\begin{aligned} |U'(p)| &= \left| (p\psi(p^2))' \frac{f(p)}{(f(p) + p)^2} + p\psi(p^2) \frac{p^2 - pf(p) - 2f(p)^2}{f(p)(f(p) + p)^3} \right| \\ &\leq C_1 \left( \frac{1}{f(p)} + \frac{2p^2 + 4pf(p) + 2f(p)^2}{f(p)(f(p) + p)^3} \right) \leq C_1 \left( \frac{1}{f(p)} + \frac{2}{f(p)^2} \right)^2 \leq C_1 \left( \frac{1}{\sqrt{\alpha}} + \frac{2}{\alpha} \right) \end{aligned}$$

and in a similar manner

$$\begin{aligned} |U''(p)| &= \left| (p\psi(p^2))'' \frac{f(p)}{(f(p) + p)^2} + 2(p\psi(p^2))' \frac{p^2 - pf(p) - 2f(p)^2}{f(p)(f(p) + p)^3} \right. \\ &\quad \left. + p\psi(p^2) \frac{5f(p)^4 + 8pf(p)^3 - 4p^3f(p) - p^4}{f(p)^3(f(p) + p)^4} \right| \\ &\leq C_2 \left( \frac{1}{\sqrt{\alpha}} + \frac{4}{\alpha} + \frac{5}{\alpha^{3/2}} \right). \end{aligned}$$

□

*Remark 4.2.* i) In the situation of Theorem 4.1, we have

$$|H(t, x, u_0)| \leq \sqrt{2\pi}(a_2 + \beta)^{3/4} \frac{\sqrt{\beta}}{\sqrt{a_2}\sqrt{a_2 - a_1 + \beta}} \sim \sqrt{2\pi\beta} a_2^{-1/4} \quad \text{as } a_2 \rightarrow +\infty.$$

ii) Suppose that  $\psi(\mu) \geq m < 0$  for  $\mu \in [\alpha', \beta']$  with  $\alpha < \alpha' < \beta' < \beta$ . Then one can show that

$$|H(t, x, u_0)| \geq \sqrt{2\pi\alpha} a_2^{-1/4} m.$$

for  $(t, x)$  satisfying

$$\sqrt{\frac{a_2 + \beta'}{\beta'}} \leq \frac{t}{x} \leq \sqrt{\frac{a_2 + \alpha'}{\alpha'}}$$

if  $a_2$  is sufficiently large. Thus the coefficient of  $t^{-1/2}$  behaves exactly as  $\text{const} \cdot a_2^{-1/4}$  (in particular it tends to zero) as  $a_2 \rightarrow +\infty$ .

iii) The cone in the  $(t, x)$ -plane, where  $u_+$  decays as  $\text{const} \cdot t^{-1/2}$  is given by

$$\sqrt{\frac{\beta}{a_2 + \beta}} \leq \frac{x}{t} \leq \sqrt{\frac{\alpha}{a_2 + \alpha}}.$$

Clearly it shrinks and inclines toward the  $t$ -axis as  $a_2 \rightarrow +\infty$ . One can prove that outside this cone,  $u_+$  decays at least as  $t^{-1}$ . This exact asymptotic behavior of the  $L^\infty$ -norm might be experimentally verified.

iv) Note that (4) also implies that

$$\begin{aligned} |u_+(t, x)| &\leq |u_+(t, x) - H(t, x, u_0)t^{-1/2} + H(t, x, u_0)t^{-1/2}| \\ &\leq C(\psi, \alpha, \beta)t^{-1} + |H(t, x, u_0)|t^{-1/2} \\ &\leq D(\psi, \beta, a_1, a_2)t^{-1/2}, \quad x \in \mathbb{R}, \quad t \geq 1. \end{aligned}$$

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